

Tal-Ajah Distribution and Its Applications

Tal Mark Pokalas^a, Samuel Ugochukwu Enogwe^b, Chisimkwuo John^c, Kingsley Uchendu^d and Ajah Chinedu Ukpai^e

^{a,b,c,d,e} *Department of Statistics, Michael Okpara University of Agriculture, Umudike, Nigeria.*

ARTICLE HISTORY

Compiled 20 DEC 2025

ABSTRACT

In this paper, a one-parameter lifetime distribution called the Tal-Ajah distribution for modeling lifetime data is proposed. Mathematical and statistical properties of the new distribution, including its survival function, hazard rate function, shape characteristics of the density, stochastic ordering, entropy measure, and stress-strength reliability, are studied. Two datasets, including a process dataset and a 40-blood cancer patient dataset, are used for the comparative study. The estimation of parameters is carried out using the method of maximum likelihood. The goodness of fit of the model is assessed using HQIC, BIC, CAIC, and AIC. The proposed distribution is compared with the Exponential, Lindley, Ishita, Akash, Pranav, Christ-Jerry, Shanker, and Rama distributions, and it shows superiority over the competing models.

KEYWORDS

Tal-Ajah distribution; Moments; Goodness of fit; Hazard rate function; Stochastic Ordering

1. Introduction

The importance of modeling lifetime data in very sensitive areas of life like medicine, biological science and industry cannot be over emphasized. Practically speaking not, all real-life data set can be well-fitted to standard probability distributions, such situation necessitates the development of new probability distributions which gives a better fit than the conventional distributions. There are several methods for developing new probability distributions, which includes the exponentiated method, weighted method, Alpha power method, mixture method, inverse transformation method etc. The earliest one parameter distributions are the Exponential and Lindley distributions.

The Lindley distribution was first proposed by [1] as a one scale parameter distribution for modelling lifetime data. A detailed study about its various mathematical properties and estimation reporting the superiority of the Lindley distribution over exponential distribution was conducted by [2]. A breakthrough that resulted to the modifications of these methods started with the work of [3] who conducted a

comparative study and reported that there are many lifetime data where these two conventional distributions cannot be suitable due to the nature of its shapes, hazard rate functions and mean residual time.

The concept of mixture method was first used by a renowned biometrician [4] who fitted a proportional mixture of two normal probability density function with different means μ_1 and μ_2 and different variable σ_1^2 and σ_2^2 based on the mixing proportion π_1 and π_2 . Thereafter, numerous one parameter distributions were proposed mostly using the mixture method.

Lindley [1] proposed a distribution that is a mixture of an exponential distribution with scale parameter θ and gamma distribution with shape parameter 2 and scale parameter θ , based on the mixing proportion $\frac{\theta}{\theta+1}$ and $\frac{1}{\theta+1}$. Shanker [5] used the mixture method to proposed another distribution called Akash distribution for modeling real-lifetime data set as a mixture of exponential distribution with scale parameter θ and gamma distribution with scale parameter θ and shape parameter 2 with a mixing proportion. Shanker [6] proposed a new distribution called Shanker distribution, which is the mixture of exponential distribution with scale parameter θ and gamma distribution with scale parameter θ and shape parameter 2, based on the mixing proportion $\frac{\theta^2}{\theta^2+1}$ and $\frac{1}{\theta^2+1}$.

Shanker and Shukla [7] also worked on a new continuous distribution called Ishita distribution, which is a two-component mixture of exponential distribution with scale parameter θ and gamma distribution with scale parameter θ and shape parameter 3, based on the mixing proportion $\frac{\theta^3}{\theta^3+2}$ and $\frac{2}{\theta^3+2}$. Shanker [8] compounded two distributions using the mixture method which he called Rama distribution, an exponential distribution with scale parameter θ , a gamma distribution with shape parameter 4 and scale parameter θ , based on the mixing proportion $\frac{\theta^3}{\theta^3+6}$ and $\frac{6}{\theta^3+6}$. Shukla [9] proposed a new continuous distribution called Pranav distribution which is a mixture of two distributions, an exponential distribution with scale parameter and gamma distribution having shape parameter 4 and scale parameter, based on the mixing proportions. Each of this lifetime probability distributions exhibit superiority over others due to their differences in hazard rate function, mean residual life function and its shape.

In most practical situations, some of these well-known statistical distributions do not fit some real-life data adequately. To make up to this short fall, a new one parameter probability distribution that is flexible and give a better fit in modeling lifetime data set than other lifetime distributions like: Exponential, lindley, Ishita, Akash, Pranav, Christ-Jerry Shanker and Rama was proposed. The proposed distribution is a compound distribution obtained by mixing an exponential distribution with scale parameter θ , a gamma distribution with scale and shape parameter θ and 3 respectively and a gamma distribution with scale parameter θ and shape parameter 5, based on the mixing proportion of $\frac{\theta^4}{\theta^4+2\theta+24}$, $\frac{2\theta}{\theta^4+2\theta+24}$ and $\frac{24}{\theta^4+2\theta+24}$ the distribution has been compared with other one parameter distributions that exhibit the same characteristics.

2. Tal-Ajah Probability Distribution

Theorem 2.1. A random variable X is said to follow the Tal-Ajah distribution (NMPD) with parameter θ , denoted by $X \sim NMPD(\theta)$, if its probability density function (pdf) is given by:

$$f_{NMPD}(x, \theta) = \frac{\theta^4}{\theta^4 + 2\theta + 24}(\theta x^4 + x^2 + \theta)e^{-\theta x}, x > 0, \theta > 0 \quad (2.1)$$

Proof. Consider $X \sim Exp(\theta)$, then

$$f_1(x, \theta) = \theta e^{-\theta x}, x > 0, \theta > 0 \quad (2.2)$$

$X \sim Gamma(3, \theta)$, then

$$f_2(x, \theta) = \frac{\theta^3 x^2 e^{-x\theta}}{2}, x > 0, \theta > 0 \quad (2.3)$$

$X \sim Gamma(5, \theta)$, then

$$f_3(x, \theta) = \frac{\theta^5 x^4 e^{-x\theta}}{24}, x > 0, \theta > 0 \quad (2.4)$$

based on the mixing propotion as $P_1 = \frac{\theta^4}{\theta^4 + 2\theta + 24}$, $P_2 = \frac{2\theta}{\theta^4 + 2\theta + 24}$, $P_3 = \frac{24}{\theta^4 + 2\theta + 24}$

By mixture method of deriving a probability distribution (pdf), the new Mixture Probability distribution is obtained from the following relation:

$$f_{NMPD}(x, \theta) = P_1 f_1(x, \theta) + P_2 f_2(x, \theta) + P_3 f_3(x, \theta), x \in \dots \quad (2.5)$$

Substituting (2.2), (2.3) and (2.4) into (2.5), we get:

$$f_{NMPD}(x, \theta) = \frac{\theta^4}{\theta^4 + 2\theta + 24} [\theta e^{-\theta x}] + \frac{2\theta}{\theta^4 + 2\theta + 24} \left[\frac{2\theta}{\theta^4 + 2\theta + 24} \right] + \frac{24}{\theta^4 + 2\theta + 24} \left[\frac{\theta^5 x^4 e^{-\theta x}}{24} \right],$$

$$x > 0, \theta > 0$$

$$f_{NMPD}(x, \theta) = \frac{\theta^4}{\theta^4 + 2\theta + 24}(\theta x^4 + x^2 + \theta)e^{-\theta x}, x > 0, \theta > 0 \quad (2.6)$$

□

Corollary 2.2. The pdf of the new mixture Probability distribution is a proper density function.

Proof. Corollary 2.2 suffices to show that the pdf in (2.1) satisfies the conditions:

$$f_{NMPD}(x, \theta) \geq 0 \quad (2.7)$$

$$\int_0^{\infty} f_{NMPD}(x, \theta) dx = 1 \tag{2.8}$$

Verifying, it is observed that from (2.1) that $f_{NMPD}(x, \theta) \geq 0$, for all $x > 0$. Also,

$$\int_0^{\infty} f_{NMPD}(x, \theta) dx = \int_0^{\infty} \frac{\theta^4}{\theta^4 + 2\theta + 24} \left[\theta x^4 + x^2 + \theta \right] e^{-\theta x} dx \tag{2.9}$$

$$\frac{\theta^4}{\theta^4 + 2\theta + 24} \int_0^{\infty} \theta x^{5-1} e^{-\theta x} dx + \frac{\theta^4}{\theta^4 + 2\theta + 24} \int_0^{\infty} \theta x^{3-1} e^{-\theta x} dx + \frac{\theta^5}{\theta^4 + 2\theta + 24} \int_0^{\infty} \theta x^{1-1} e^{-\theta x} dx$$

$$\frac{\theta^4}{\theta^4 + 2\theta + 24} \left[\frac{24}{\theta^4} + \frac{2}{\theta^3} + \frac{\theta}{\theta} \right] = 1 \tag{2.10}$$

□

Theorem 2.3. *A random variable X is said to follow the Tal-Ajah probability distribution with parameter θ , denoted by $X \sim NMPD(\theta)$, if its cumulative function is given by:*

$$F_{NMPD}(x) = 1 - \left[1 + \frac{\theta^2 x(2 + \theta x)}{\theta^4 + 2\theta + 24} + \frac{\theta x(\theta^3 x^3 + 4\theta^2 x^2 + 12\theta x + 24)}{\theta^4 + 2\theta + 24} \right] e^{-\theta x}, x > 0, \theta > 0 \tag{2.11}$$

Proof. Let $X \sim Exp$, then

$$F_1(x) = 1 - e^{-\theta x}, x > 0, \theta > 0 \tag{2.12}$$

$X \sim Gamma(3, \theta)$, then

$$F_2(x) = 1 - \left[1 + \frac{\theta x(2 + \theta x)}{2} \right] e^{-\theta x}, x > 0, \theta > 0 \tag{2.13}$$

$X \sim Gamma(5, \theta)$, then

$$F_3(x) = 1 - \left[1 + \frac{\theta x(\theta^3 x^3 + 4x^2\theta^2 + 12x\theta + 24)}{24} \right] e^{-\theta x}, x > 0, \theta > 0 \tag{2.14}$$

Based on the mixing propotion as $P_1 = \frac{\theta^4}{\theta^4 + 2\theta + 24}$, $P_2 = \frac{2\theta}{\theta^4 + 2\theta + 24}$, $P_3 = \frac{24}{\theta^4 + 2\theta + 24}$

By mixture method of deriving a cumulative distribution function (cdf), the New Mixture Probability distribution is obtained from the following relation:

$$F_{NMPD}(x, \theta) = P_1F_1(x, \theta) + P_2F_2(x, \theta) + P_3F_3(x, \theta), x \in \dots \quad (2.15)$$

Substituting (2.12), (2.13) and (2.14) into (2.15), gives

$$F_{NMPD}(x) = 1 - \left[1 + \frac{\theta^2x(2+\theta x)}{\theta^4+2\theta+24} + \frac{\theta x(\theta^3x^3+4\theta^2x^2+12\theta x+24)}{\theta^4+2\theta+24} \right] e^{-\theta x}, x > 0, \theta > 0$$

□

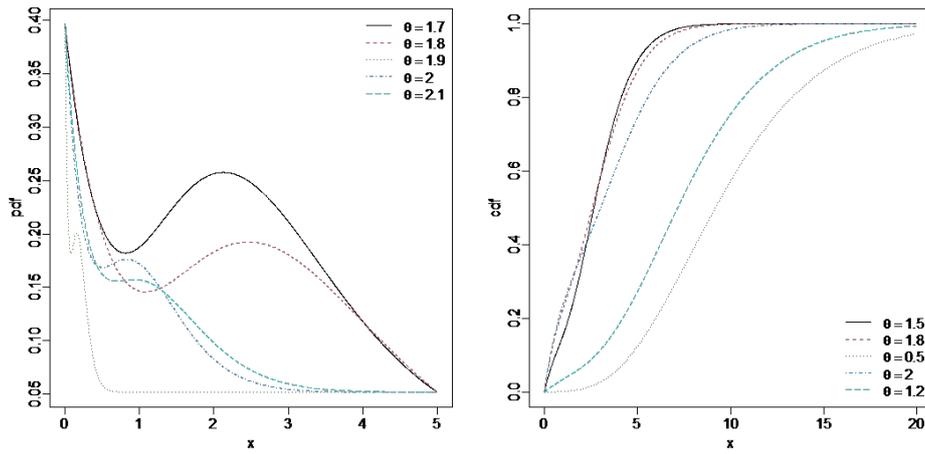
Corollary 2.4. *The cdf in (2.11) satisfies the conditions $F_{NMPD}(-\infty) = 0$ and $F_{NMPD}(\infty) = 1$.*

Proof. The proof of corollary 2.4 follows from taking the limit of the (c.d.f) in (2.11) as $X \rightarrow -\infty$ and $X \rightarrow \infty$. Thus,

$$F(-\infty) = \lim_{x \rightarrow -\infty} F(x, \theta) = \lim_{x \rightarrow -\infty} \left(1 - \left[1 + \frac{\theta^2x(2+\theta x)}{\theta^4+2\theta+24} + \frac{\theta x(\theta^3x^3+4\theta^2x^2+12\theta x+24)}{\theta^4+2\theta+24} \right] e^{-\theta x} \right) = 0 \quad (2.16)$$

$$F(\infty) = \lim_{x \rightarrow \infty} F(x, \theta) = \lim_{x \rightarrow -\infty} \left(1 - \left[1 + \frac{\theta^2x(2+\theta x)}{\theta^4+2\theta+24} + \frac{\theta x(\theta^3x^3+4\theta^2x^2+12\theta x+24)}{\theta^4+2\theta+24} \right] e^{-\theta x} \right) = 1 \quad (2.17)$$

□



(a) Graph of the PDF of Tal-Ajah distribution for varying values of the parameter θ . (b) Graph of the CDF of Tal-Ajah distribution for varying values of the parameter θ .

Figure 1. The graph of the p.d.f and the c.d.f of the Tal-Ajah distribution for various values of the parameter θ .

2.1. Crude Moments of the Tal-Ajah Probability Distribution

Theorem 2.5. Given that $X \sim NMPD(\theta)$, the r^{th} crude moment of X is given by:

$$\mu_r^1 = \frac{r! \left[\theta^4 + \theta(r+2)(r+1) + (r+4)(r+3)(r+2)(r+1) \right]}{\theta^r(\theta^4 + 2\theta + 24)}, r = 1, 2, 3, 4, \dots \quad (2.18)$$

Corollary 2.6. If $r = 1, 2, 3, 4$ and substituted into (2.18) will yield the first four crude moments of a random variable $X \sim NMPD(\theta)$

$$\mu_1^1 = \frac{1! \left[\theta^4 + \theta(1+2)(1+1) + (1+4)(1+3)(1+2)(1+1) \right]}{\theta^1(\theta^4 + 2\theta + 24)} = \frac{\theta^4 + 6\theta + 120}{\theta(\theta^4 + 2\theta + 24)} \quad (2.19)$$

$$\mu_2^1 = \frac{2! \left[\theta^4 + \theta(2+2)(2+1) + (2+4)(2+3)(2+2)(2+1) \right]}{\theta^2(\theta^4 + 2\theta + 24)} = \frac{2(\theta^4 + 12\theta + 360)}{\theta^2(\theta^4 + 2\theta + 24)} \quad (2.20)$$

$$\mu_3^1 = \frac{3! \left[\theta^4 + \theta(3+2)(3+1) + (3+4)(3+3)(3+2)(3+1) \right]}{\theta^3(\theta^4 + 2\theta + 24)} = \frac{6(\theta^4 + 20\theta + 840)}{\theta^3(\theta^4 + 2\theta + 24)} \quad (2.21)$$

$$\mu_4^1 = \frac{4! \left[\theta^4 + \theta(4+2)(4+1) + (4+4)(4+3)(4+2)(4+1) \right]}{\theta^4(\theta^4 + 2\theta + 24)} = \frac{24(\theta^4 + 30\theta + 1680)}{\theta^4(\theta^4 + 2\theta + 24)} \quad (2.22)$$

2.2. Central Moments of the Tal-Ajah Probability Distribution

The first four central moments of $X \sim NMPD(\theta)$ are given as:

$$\mu_1 = \mu_1^1 - \mu_1^1 = \frac{\theta^4 + 6\theta + 120}{\theta(\theta^4 + 2\theta + 24)} - \frac{\theta^4 + 6\theta + 120}{\theta(\theta^4 + 2\theta + 24)} = 0 \quad (2.23)$$

$$\mu_2 = \mu_2^1 - (\mu_1^1)^2 = \frac{2(\theta^4 + 120\theta + 360)}{\theta^2(\theta^4 + 2\theta + 24)} - \left[\frac{\theta^4 + 6\theta + 120}{\theta(\theta^4 + 2\theta + 24)} \right]^2 = 0 \quad (2.24)$$

$$\mu_2 = \frac{\theta^8 + 16\theta^5 + 528\theta^4 + 12\theta^2 + 576\theta + 2880}{\theta^2(\theta^4 + 2\theta + 24)^2} \quad (2.25)$$

$$\mu_2 = \mu_2^1 - 3\mu_2^1\mu_1^1 + 2(\mu_1^1)^2 \tag{2.26}$$

$$\mu_2 = \left[\frac{2\left(\theta^{12} + 30\theta^9 + 1512\theta^8 + 36\theta^6 + 1149\theta^5 + 36288\theta^4 + 24\theta^3 + 2016\theta^2 + 19280\theta + 34560\right)}{\theta^3(\theta^4 + 20\theta + 24)^3} \right] \tag{2.27}$$

$$\mu_4 = \mu_4^1 - 4\mu_3^1\mu_1^1 + 6\mu_2^1(\mu_1^1)^2 - 3(\mu_1^1)^4 \tag{2.28}$$

$$\begin{aligned} &= \frac{24(\theta^4 + 30\theta + 1680)}{\theta^4(\theta^4 + 30\theta + 1680)} - 4 \left[\frac{6(\theta^4 + 20\theta + 840)(\theta^4 + 6\theta + 120)}{\theta^4(\theta^4 + 2\theta + 24)^2} \right] \\ &+ 6 \left[\frac{2(\theta^4 + 12\theta + 360)(\theta^4 + 6\theta + 120)^2}{\theta^2(\theta^4 + 2\theta + 24)^3\theta^2} \right] - 3 \left[\frac{(\theta^4 + 6\theta + 120)}{\theta(\theta^4 + 2\theta + 24)} \right]^4 \end{aligned} \tag{2.29}$$

$$\mu_4 = \frac{\left[\begin{aligned} &9\theta^{19} + 3192\theta^{13} + 23904\theta^{12} + 1224\theta^{10} + 96240\theta^9 + 528768\theta^8 \\ &+ 1728\theta^7 + 10117792\theta^6 + 2021760\theta^5 + 11115216\theta^4 + 10601856\theta^3 \\ &+ 334869120\theta^2 + 13271040\theta + 34836480 \end{aligned} \right]}{\theta^4(\theta^4 + 2\theta + 24)^4} \tag{2.30}$$

2.3. Skewness and Kurtosis of the Tal-Ajah Probability Distribution

Given that $X \sim NMPD(\theta)$, the coefficient of skewness for X is given by:

$$\begin{aligned} \gamma_1 &= \frac{\mu_3}{\mu_2^{\frac{3}{2}}} = \frac{2 \left[\frac{\theta^{12} + 30\theta^9 + 1512\theta^8 + 36\theta^6 + 1149\theta^5 + 36288\theta^4 + 24\theta^3 + 2016\theta^2 + 19280\theta + 24560}{\theta^3(\theta^4 + 2\theta + 24)^3} \right]}{\left[\frac{\theta^2(\theta^4 + 2\theta + 24)^2}{\theta^8 + 16\theta^5 + 528\theta^4 + 12\theta^2 + 576\theta + 2880} \right]^{\frac{3}{2}}} \\ &= \frac{2 \left[\frac{\theta^{12} + 30\theta^9 + 1512\theta^8 + 36\theta^6 + 1149\theta^5 + 36288\theta^4 + 24\theta^3 + 2016\theta^2 + 19280\theta + 24560}{\theta^3(\theta^4 + 2\theta + 24)^3} \right]}{\left[\theta^8 + 16\theta^5 + 528\theta^4 + 12\theta^2 + 576\theta + 2880 \right]^{\frac{3}{2}}} \end{aligned} \tag{2.31}$$

The coefficient of kurtosis for $X \sim NMPD(\theta)$ is given by:

$$\gamma_2 = \frac{\mu_4}{\mu_2^2} = \frac{\left[\begin{array}{c} 9\theta^{16} + 3192\theta^{13} + 23904\theta^{12} + 12240\theta^{10} + 9624\theta^9 \\ + 528768\theta^8 + 1728\theta^7 + 1017792\theta^6 + 2021760\theta^5 + 11115216\theta^4 \\ + 10601856\theta^3 + 334869120\theta^2 + 13271040\theta + 34836480 \end{array} \right]}{\left(\theta^8 + 16\theta^5 + 528\theta^4 + 12\theta^2 + 576\theta + 2880\right)^2} \quad (2.32)$$

2.4. Coefficient of Variation and Index of Dispersion

The coefficient of variation of variation (φ) for $X \sim NMPD(\theta)$ is given by:

$$\varphi = \frac{\sigma}{\mu_1} = \frac{\sqrt{\theta^8 + 16\theta^5 + 528\theta^4 + 12\theta^2 + 576\theta + 2880}}{\theta(\theta^4 + 2\theta + 24)} \quad (2.33)$$

The Index of dispersion (η) for $X \sim NMPD(\theta)$ is given by:

$$\eta = \frac{\sigma^2}{\mu_1^2} = \left[\frac{\sqrt{\theta^8 + 16\theta^5 + 528\theta^4 + 12\theta^2 + 576\theta + 2880}}{\theta(\theta^4 + 2\theta + 24)(\theta^4 + 6\theta + 120)} \right] \quad (2.34)$$

Notably, if $\mu < \sigma^2, \mu = \sigma^2, \mu > \sigma^2$, the new probability distribution is over-dispersed, equi-dispersed and under-dispersed respectively.

2.5. Moment Generating Function

The moment generating function of $X \sim NMPD(\theta)$ is given by:

$$\begin{aligned} M_X(t) &= E(e^{tX}) = \int_0^\infty e^{Xt} f_{NMPD}(x, \theta) dx = \int_0^\infty \left[\frac{\theta^4}{\theta^4 + 2\theta + 24} (\theta x^4 + x^2 + \theta) e^{-x\theta} dx \right] \\ &= \frac{\theta^4}{\theta^4 + 2\theta + 24} \left[\int_0^\infty \theta x^{5-1} e^{-x(\theta-t)} dx + \int_0^\infty \theta x^{3-1} e^{-x(\theta-t)} dx + \int_0^\infty \theta x^{1-1} e^{-x(\theta-t)} dx \right] \\ &= \frac{\theta^4}{\theta^4 + 2\theta + 24} \left[\frac{\theta \Gamma(5)}{(\theta - t)^5} + \frac{\Gamma(3)}{(\theta - t)^3} + \frac{\theta \Gamma(1)}{(\theta - t)^1} \right] \\ &= \frac{\theta^4}{\theta^4 + 2\theta + 24} \left[\frac{24\theta}{(\theta - t)^5} + \frac{2}{(\theta - t)^3} + \frac{\theta}{(\theta - t)^1} \right] \\ &= \frac{1}{\theta^4 + 2\theta + 24} \left[24 \left(1 - \frac{t}{\theta}\right)^{-5} + 2\theta \left(1 - \frac{t}{\theta}\right)^{-3} + \theta^4 \left(1 - \frac{t}{\theta}\right)^{-1} \right] \end{aligned} \quad (2.35)$$

Using Binomial expansion,

$$(1 - x)^k = \sum_{k=0}^\infty \binom{k+r-1}{k} x^k; r > 0 \quad \text{and} \quad |x| < 1 \quad (2.36)$$

$$M_X(X) = \frac{1}{\theta^4 + 2\theta + 24} \left[\theta^4 \sum_{k=0}^{\infty} \binom{t}{\theta}^k + 2\theta \sum_{k=0}^{\infty} \binom{k+2}{k} \binom{t}{\theta}^k + 24 \sum_{k=0}^{\infty} \binom{k+4}{k} \binom{t}{\theta}^k \right] \tag{2.37}$$

$$= \sum_{k=0}^{\infty} \left[\frac{(k+4) + (k+3) + (k+2) + (k+1)}{\theta^4} + \frac{(k+2) + (k+1)}{\theta^3} + 1 \right] \binom{t}{\theta}^k$$

$$M_X(t) = \frac{\sum_{k=0}^{\infty} \left[\theta^4 + \theta(k+2) + (k+1) + (k+4)(k+3)(k+2) + (k+1) \right] \binom{t}{\theta}^k}{\theta^4 + 2\theta + 24} \tag{2.38}$$

2.6. Survival, Hazard Rate Function and Mean Residual Life Function of Tal-Ajah Distribution

The Survival function of $X \sim NMPD(\theta)$ is given by:

$$S(x) = 1 - F_{NMPD}(x, \theta)$$

$$S(x) = 1 - \left[1 - \left(1 + \frac{\theta^2 x(2 + x\theta) + \theta x(x^3\theta^3 + 4x^2\theta^2 + 12x\theta + 24)}{\theta^4 + 2\theta + 24} \right) e^{-x\theta} \right] \tag{2.39}$$

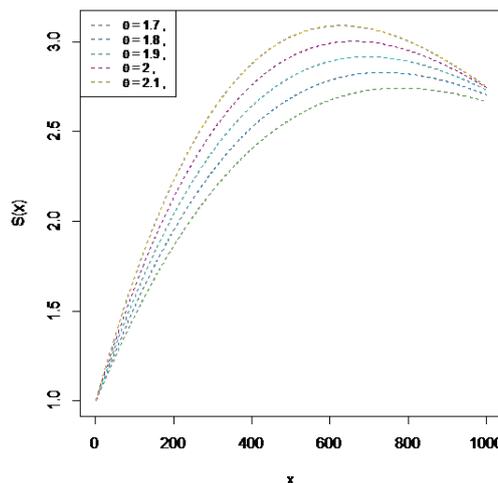


Figure 2. Graph of Survival rate function of Tal-Ajah distribution for varying values of the parameter α .

The hazard rate function of $X \sim NMPD(\theta)$ is given by:

$$h(x; \theta) = \frac{f_{NMPD}(x; \theta)}{1 - F_{NMPD}(x; \theta)} \tag{2.40}$$

$$= \frac{\frac{\theta^4}{\theta^4 + 2\theta + 24}(\theta + x^2 + \theta x^4)e^{-x\theta}}{\left[1 + \frac{\theta^2 x + (2+x\theta) + \theta x(x^3\theta^3 + 4x^2\theta^2 + 12x\theta + 24)}{\theta^4 + 2\theta + 24} \right]}$$

$$h(x; \theta) = \frac{\theta^5 + \theta^4 x + \theta^5 x^4}{\theta^4 + 2\theta + 24 + \theta^2 x(2 + x\theta) + \theta x(x^3\theta^3 + 4x^2\theta^2 + 12x\theta + 24)} \tag{2.41}$$

Notably,

$$h(0) = \frac{\theta^5 + \theta^4(0) + \theta^5(0)^4}{\theta^4 + 2\theta + 24 + \theta^2(0)(2 + (0)\theta) + \theta(0)((0)^3\theta^3 + 4(0)^2\theta^2 + 12(0)\theta + 24)}$$

$$h(0) = \frac{\theta^5}{\theta^4 + 2\theta + 24} = f_{NMPD}(0) \tag{2.42}$$

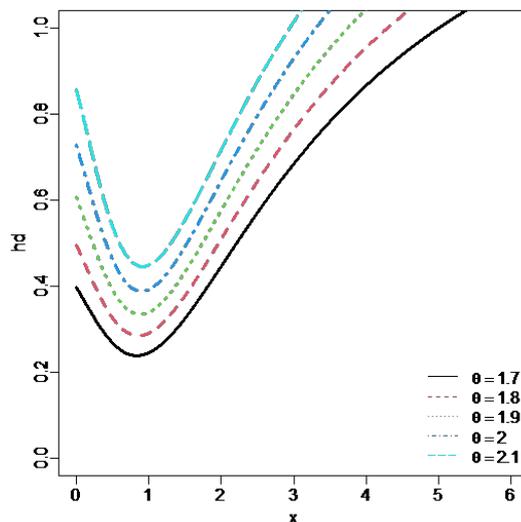


Figure 3. Graph of hazard rate function of Tal-Ajah distribution for varying values of the parameter θ .

The mean residual life function of $X \sim NMPD(\theta)$ is given by:

$$m(x) = E[X - x/X > x] = \frac{1}{1 - F_{NMPD}(x)} \int_x^\infty [1 - F_{NMPD}(t)] dt \tag{2.43}$$

$$\begin{aligned}
 m(x) &= \frac{\theta^4 x^4 + 8\theta^3 x^3 + (36\theta^2 + \theta^3)x^2 + (96\theta + 2\theta^2)x + \theta^4 + 6\theta^2 + 120}{\theta[\theta^4 x^4 + 4\theta^3 x^3 + (12\theta^2 + \theta^3)x^2 + (24\theta + 2\theta^2)x + (\theta^4 + 2\theta + 24)]} \\
 m(0) &= \frac{\theta^4(0)^4 + 8\theta^3(0)^3 + (36\theta^2 + \theta^3)(0)^2 + (96\theta + 2\theta^2)(0) + \theta^4 + 6\theta^2 + 120}{\theta[\theta^4(0)^4 + 4\theta^3(0)^3 + (12\theta^2 + \theta^3)(0)^2 + (24\theta + 2\theta^2)(0) + (\theta^4 + 2\theta + 24)]} \\
 &= \frac{\theta^4 + 6\theta + 120}{\theta^4 + 2\theta + 24} = \mu \tag{2.44}
 \end{aligned}$$

2.7. Stochastic Ordering

Theorem 2.7. Let $X \sim NMPD(\theta_1)$ and $Y \sim NMPD(\theta_2)$. If $\theta_1 > \theta_2$, then $X \leq_{lor} Y$ and $X \leq_{Ror} Y$, $X \leq_{orl} Y$ and $X \leq_{stor} Y$.

Proof. The relation of the distribution of $X \sim NMPD(\theta_1)$ to the distribution of $Y \sim NMPD(\theta_2)$ is given by:

$$\frac{f_X(x, \theta_1)}{f_Y(x, \theta_2)} = \frac{\theta_1^4}{\theta_1^4 + 2\theta_1 + 24} (\theta_1 x^4 + x^2 + \theta_1) e^{-\theta_1 x} * \frac{\theta_2^4 + 2\theta_2 + 24}{\theta_2^4 (\theta_2 x^4 + x^2 + \theta_2) e^{-\theta_2 x}} \tag{2.45}$$

$$\frac{f_X(x, \theta_1)}{f_Y(x, \theta_2)} = \frac{\theta_1^4 (\theta_1 x^4 + x^2 + \theta_1) (\theta_2^4 + 2\theta_2 + 24) e^{-\theta_1 x}}{\theta_2^4 (\theta_2 x^4 + x^2 + \theta_2) (\theta_1^4 + 2\theta_1 + 24) e^{-\theta_2 x}}$$

$$\frac{f_X(x, \theta_1)}{f_Y(x, \theta_2)} = \frac{\theta_1^4 (\theta_1 x^4 + x^2 + \theta_1) (\theta_2^4 + 2\theta_2 + 24)}{\theta_2^4 (\theta_2 x^4 + x^2 + \theta_2) (\theta_1^4 + 2\theta_1 + 24)} e^{-(\theta_1 - \theta_2)x}, x > 0 \tag{2.46}$$

Taking the natural logarithm of (2.46) leads to

$$\ln \frac{f_X(x, \theta_1)}{f_Y(x, \theta_2)} = \ln \left[\frac{\theta_1^4 (\theta_1 x^4 + x^2 + \theta_1) (\theta_2^4 + 2\theta_2 + 24)}{\theta_2^4 (\theta_2 x^4 + x^2 + \theta_2) (\theta_1^4 + 2\theta_1 + 24)} e^{-(\theta_1 - \theta_2)x} \right], x > 0 \tag{2.47}$$

$$\ln \frac{f_X(x, \theta_1)}{f_Y(x, \theta_2)} = \ln \left[\frac{\theta_1^4 (\theta_2^4 + 2\theta_2 + 24)}{\theta_2^4 (\theta_1^4 + 2\theta_1 + 24)} \right] + \ln [\theta_1^4 (\theta_1 x^4 + x^2 + \theta_1)] - \ln [\theta_2^4 (\theta_2 x^4 + x^2 + \theta_2)] - (\theta_1 - \theta_2)x \tag{2.48}$$

Differentiating (2.48) with respect to x gives:

$$\frac{d}{dx} \ln \frac{f_X(x, \theta_1)}{f_Y(x, \theta_2)} = \frac{(4x^3 \theta_1 + 2x)(\theta_2 x^4 + x^2 + \theta_2) - (4x^3 \theta_2 + 2x)(\theta_1 x^4 + x^2 + \theta_2)}{(\theta_1 x^4 + x^2 + \theta_1)(\theta_2 x^4 + x^2 + \theta_2)} - (\theta_2 - \theta_1) \tag{2.49}$$

This implies that $\frac{d}{dx} \ln \frac{f_X(x, \theta_1)}{f_Y(x, \theta_2)} < 0$. If $\theta_1 > \theta_2$, then $X \leq_{Ror} Y$, $X \leq_{orl} Y$ and $X \leq_{stor} Y$.

□

3. Estimation of Parameter

3.1. Maximum Likelihood Estimation Method:

In estimation theory, the Method of Maximum Likelihood have been used more often to find the parameters of statistical distributions because of its consistently, asymptotic, efficiently and in variance properties. Thus, to obtain the maximum likelihood estimator of the Tal-Ajah probability distribution. Let $x_1, x_2, x_3, \dots, x_n$ constitute a random sample of size n from this distribution and define the likelihood function of the random sample as:

$$L = \prod_{i=1}^n f_{NMPD}(x, \theta) \tag{3.1}$$

$$= \prod_{i=1}^n \left[\frac{\theta^4}{\theta^4 + 2\theta + 24} (\theta x_i^4 + x_i^2 + \theta) e^{-\theta x_i} \right] \tag{3.2}$$

$$= \left(\frac{\theta^4}{\theta^4 + 2\theta + 24} \right)^n \prod_{i=1}^n [(\theta x_i^4 + x_i^2 + \theta) e^{-\theta x_i}]$$

$$= \left(\frac{\theta^4}{\theta^4 + 2\theta + 24} \right)^n \prod_{i=1}^n [(\theta x_i^4 + x_i^2 + \theta) e^{-\theta \bar{x}}]$$

Taking the natural logarithm gives:

$$\ln L = n \ln [\theta^4 - (\theta^4 + 2\theta + 24)] + \sum_{i=1}^n \ln [\theta x_i^4 + x_i^2 + \theta] - n\theta \bar{x} \tag{3.3}$$

$$= n \left[4 \ln \theta - \ln(\theta^4 + 2\theta + 24) \right] + \sum_{i=1}^n \ln [\theta x_i^4 + x_i^2 + \theta] - n\theta \bar{x}$$

Differentiating (3.52) with respect to θ

$$\frac{d \ln L}{d \theta} = n \left[\frac{4}{\theta} - \frac{4\theta^3 + 2}{\theta^4 + 2\theta + 24} \right] + \sum_{i=1}^n \left[\frac{x_i^4}{\theta} + \frac{1}{\theta} \right] - n\bar{x} = 0 \tag{3.4}$$

$$= \frac{4n}{\theta} - \frac{4n\theta^3 + 2}{\theta^4 + 2\theta + 24} + \sum_{i=1}^n \left[\frac{x_i^4}{\theta} + \frac{1}{\theta} \right] - n\bar{x} = 0 \quad (3.5)$$

4. Goodness of Fit

The Tal-Ajah distribution was applied to two real life data sets was used for the comparison using the maximum likelihood estimation and method of moment order to ascertain its superiority over eight probability distribution models compared with such as: Exponential, Lindley, Ishita, Akash, Pranav, Chris-Jerry, Shanker and Rama distributions. The results were compared with the other competing models using the various Goodness of fit indices which include Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), K-S Statistics (Kolmogorov-Smirnov Statistics) and corrected Akaike Information Criterion (AIC). Their computations formula are given below:

$$AIC = -2\ln L + 2K \quad (4.1)$$

$$BIC = -2\ln L + K\log(n) \quad (4.2)$$

$$CAIC = AIC + \frac{2K(K + 1)}{(n - k - 1)} \quad (4.3)$$

Where k = Number of parameters, S = Sample Size. The two data sets are shown below as follows:

Data set 1:

The data set represent 40 patients suffering from blood cancer (leukemia) from one ministry of health hospitals in Saudi Arabia as reported by Abouammoh et al. [10] The ordered lifetimes data (in years) are given below:

0.315, 0.496, 0.616, 1.145, 1.208, 1.263, 1.414, 2.025, 2.036, 2.162, 2.211, 2.37, 2.532, 2.693, 2.805, 2.91, 2.912, 3.192, 3.263, 3.348, 3.427, 3.499, 3.534, 3.767, 3.858, 3.751, 3.858, 3.986, 4.049, 4.244, 4.323, 4.381, 4.397, 4.647, 4.753, 4.929, 4.973, 5.074, 5.381

Table 1. The Crude Moments, Variance, Coefficient of Variation, Skewness, Kurtosis and Coefficient of Dispersion.

θ	μ_1^1	μ_1^2	μ_1^3	μ_1^4	σ^2	φ	γ_1	γ_2	$\eta = \frac{\sigma^2}{\mu_1^1}$
0.1	49.8346	2985.112	208759.5	1669084	501.629	22.39711	0.4586	4.5795	10.066
0.3071	16.1142	313.2305	7118.967	185112.6	53.5634	7.318703	0.4837	7.558867	3.3240
0.5143	9.5409	110.3356	1494.354	23173.62	19.3062	4.393954	0.5218	12.80848	2.0235
0.7214	6.7170	55.1386	531.2176	5865.169	10.0204	3.165499	0.5897	19.03147	1.4918
0.9286	5.1137	32.4257	242.0605	2073.21	6.2760	2.505208	0.6994	24.21323	1.2273
1.1357	4.0496	20.8316	126.7403	886.0492	4.4321	2.105256	0.8457	26.31825	1.0944
1.3429	3.2661	14.0533	72.00654	424.8496	3.3859	1.840081	1.0069	24.74518	1.0367
1.5500	2.6509	9.734018	42.97507	219.1161	2.7063	1.645081	1.1606	20.74626	1.0209
1.7571	2.1519	6.830371	26.41198	118.4106	2.1998	1.483161	1.2980	16.2154	1.0223
1.9643	1.7439	4.823786	16.53253	66.0319	1.7824	1.335076	1.4237	12.43538	1.0220
2.1714	1.4138	3.424451	10.49472	37.72718	1.4256	1.193977	1.5461	9.850156	1.0083
2.3786	1.1504	2.446464	6.746962	22.00378	1.1230	1.059724	1.6712	8.400859	0.9762
2.5857	0.9435	1.764456	4.399523	13.09904	0.8743	0.935041	1.8006	7.867664	0.9267
2.7929	0.7824	1.288263	2.913721	7.958858	0.6761	0.822273	1.9332	8.023049	0.8642
3.0000	0.6577	0.954955	1.963964	4.940941	0.5224	0.722801	2.0651	8.678770	0.7943

Values of first four crude moments, variance (σ^2), coefficient of skewness (γ_1), coefficient of kurtosis (γ_2), coefficient of variation (φ), and coefficient of dispersion (η) of the Tal-Ajah probability distribution over the range of 0.1 – 3 is given in table 1 above.

Table 2. Estimates of model parameter with standard errors and corresponding values of model selection criteria for the distributions fitted to data set 1

Model	Parameter Estimate	AIC	CAIC	BIC	HQIC	SE
Tal-Ajah	1.38840	141.4275	141.5356	143.5590	142.0243	0.09108
Exponential	0.32030	168.7939	168.9020	170.4575	169.3908	0.05129
Lindley	0.52980	158.1467	158.2549	159.8103	158.7436	0.06187
Ishita	0.80890	149.6288	149.7369	151.2924	150.2257	0.06534
Akash	0.80430	150.6880	150.7961	152.3516	151.2849	0.07145
Pranav	1.08440	145.2801	145.3882	146.9437	145.8770	0.07232
Chris-Jerry	0.80625	152.0595	152.1676	153.7230	152.6564	0.07990
Shanker	0.55160	155.5950	155.7032	157.2586	156.1919	0.05825
Rama	1.10498	144.9043	145.0124	146.5679	145.5012	0.08080

From **Table 2** above, the Tal-Ajah distribution has the lowest AIC value of 141.4275, the lowest BIC value of 143.559, the lowest HQIC value of 142.0243, and the lowest CAIC value of 141.5356 therefore, the Tal-Ajah distribution provides a better fit than the Exponential, Lindley, Ishita, Akash, Pranav, Chris-Jerry, Shanker and Rama distributions when fitted to data set 1.

5. Conclusion

This study developed a new lifetime continuous probability distribution with one parameter called Tal-Ajah distribution for modeling life time data. The distribution was derived using mixture method. Statistical properties like: probability density, cumulative density function, survival rate function, skewness, kurtosis, stress strength reliability, stochastic ordering and the hazard rate function of the distribution were successfully derived.

The maximum likelihood estimation and method of moments of the new mixture probability distribution were discussed for parameter estimation. Lastly, the goodness of fit test has been applied to some lifetime data set which was used to demonstrate the superiority of the proposed probability distribution other the other competing distributions. The proposed distribution yields a satisfactory result from the comparative

study. The proposed distribution has a uni-modal shape and it can be used in modeling data set that are positively skewed and decay exponentially.

Acknowledgements

This research did not receive any specific form of grant or financial assistance from any funding agency be it government or public. The authors are grateful to God Almighty for his grace. We are also grateful to the editor-in-chief and the unknown reviewers whose comments has improved the quality of this paper.

References

- [1] Lindley, D. V. (1958). Fiducial distribution and Bayes' theorem. *Journal of the Royal Statistical Society*, **20**(1), 102-107.
- [2] Ghitany, M. E., Atieh, B., & Nadarajah, S. (2008). Lindley distribution and its applications. *Mathematics Computing and Simulation*, **78**(1), 493-506.
- [3] Shanker, R., Hagos, F., & Sujatha, S. (2015). On modelling of lifetime data using exponential and Lindley distributions. *Biometrics and Biostatistics International Journal*, **2**(5), 1-9.
- [4] Pearson, K. (1894). Contributions to the mathematical theory of evolution. *Philosophical Transactions of the Royal Society of London*, **185**, 75-110.
- [5] Shanker, R. (2015a). Akash distribution and its applications. *International Journal of Probability and Statistics*, **4**(3), 65-75.
- [6] Shanker, R. (2015b). Shanker distribution and its application. *International Journal of Statistics and Applications*, **5**(6), 338-348.
- [7] Shanker, R., & Shukla, K. K. (2017). Ishita distribution and its applications. *International Journal of Biometrics and Biostatistics*, **5**(2), 00126.
- [8] Shanker, R. (2017). Rama distribution and its application. *International Journal of Statistics and Applications*, **7**(1), 26-35.
- [9] Shukla, K. K. (2018). Pranav distribution with properties and applications. *International Journal of Biometrics and Biostatistics*, **7**(3), 244-254.
- [10] Abouammoh, A., Abdulghani, S., & Qamber, I. (1994). On partial orderings and testing of new better than renewable used classes. *Reliability Engineering and System Safety*, **43**(1), 37-41.